Algebra III

Back Paper Examination

Instructions: All rings are assumed to be commutative with unity. All questions carry equal marks.

1. Let $J \subset I \subset R$ be two ideals in a ring R. Let S denote the ring R/J and $\bar{I}$ be the image of I in S under quotient map. Prove that the rings R/J and $S/\bar{I}$ are isomorphic (The third isomorphism theorem).

2. Let a be an element of a ring R and let R' be the ring obtained by adjoining an inverse of a. If α denote the inverse of a in R', prove that every element β of R' can be written as $\beta = \alpha^k \cdot b$ for some non-negative integer k and $b \in R$.

3. Let I, J be ideals in a ring R. Define the ideals $I \cap J$ and IJ of R. Prove that $IJ \subset I \cap J$. Give an example where this inclusion is a strict inclusion.

4. Let $\phi : \mathbb{R}[X] \to \mathbb{C} \times \mathbb{C}$ be the ring homomorphism defined by sending X to (i, i) and r to (r, r) for all $r \in \mathbb{R}$. Prove that the kernel of ϕ is principal and find its generator.

5. Define prime and irreducible elements of an integral domain. If R is a principal ideal domain, then prove that an element is prime if and only if it is irreducible.

6. Define primitive elements in the ring $\mathbb{Z}[X]$. Prove that the product of primitive elements is a primitive element in $\mathbb{Z}[X]$.

7. Define noetherian rings. Prove that if R is a noetherian ring, then the polynomial ring $R[X]$ is noetherian.

8. Prove that a submodule of $\mathbb{Z}^n$ is free of rank $r \leq n$.